

Waves, analytical signals, and some postulates of quantum theory

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Abstract

In this paper we apply the formalism of the analytical signal theory to the Schrödinger wavefunction. Making use exclusively of the wave-particle duality and the principle of relativistic covariance, we actually derive the form of the quantum energy and momentum operators for a single nonrelativistic particle. Without using any more quantum postulates, and employing the formalism of the characteristic function, we also derive the quantum-mechanical prescription for the measurement probability in such cases.

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1 Introduction

This paper presents a derivation of two measurement postulates of Quantum mechanics using exclusively de Broglie's original relationship and Planck's formula as the only previous quantum postulates assumed. Namely, the associations $E \rightarrow -i\hbar\partial/\partial t$ and $p \rightarrow -i\hbar\partial/\partial x$ for (nonrelativistic) particles are proved for one-particle mechanical systems, as well as the postulated formula of the measurement probability for the two observables, both with discrete and continuous spectra.

Starting from general wave equations, a theory is constructed based on the envelope of the space-time analytical signal of a certain real-valued "auxiliary" wavefunction, modulated in a latent way by its relativistic rest energy. Apart from the relativistic covariance, in this paper we will only use the principle of wave-particle duality as the starting postulate, which is summarized by the elementary relations

$$E = \hbar\omega \tag{1}$$

$$\mathbf{p} = \hbar\mathbf{k}. \tag{2}$$

Actually, the Planck-Einstein relationship (1) can even be derived on purely relativistic grounds, with the only additional assumption of the existence of an energy quantum, but without the need of postulating the specific value $\hbar\omega$. [1]

It is usual in Optics to employ mathematical tools from the Signal Theory field, where the Fourier transforms and the representations of dual transformed spaces (space - time - spatial frequency - temporal frequency) are customarily used. The analytical signals and their paraxial equations allow to undertake the study of signals of extremely high frequencies like the optical fields. However, the mathematical formalism of the analytical signal tends to be handled in a hardly rigorous way in the literature, sometimes even leading to major errors [2]. We anticipate that a careful, accurate use of the analytical signal and related concepts will be *essential* in this paper. The notational rigor should not be underestimated in what follows.

Section 2 summarizes the necessary formalism of the Fourier transforms and analytical signals. In Section 3, the Klein-Gordon and Schrödinger equations for a particle in a potential are reviewed with the new notation. Sections 4 and 6 present the derivation of the quantum-mechanical formula for the average energy and momentum of a non-relativistic particle. Special attention is paid to the energy stationary states in Section 5. With these results, the postulate of the measurement probability is derived in Section 7. The conclusions of the work are summarized in Section 8.

2 Analytical signals and Fourier transforms in time and space domains

We need to review briefly a few concepts of the analytical signal theory, as well as introduce the notation we will use. Consider a space-time scalar "wavefunction" $\psi(\mathbf{r}, t)$. For reasons that will become clear later, $\psi(\mathbf{r}, t)$ is considered to be a *real-valued* wavefunction; consequently, $\psi(\mathbf{r}, t)$ does *not* coincide with the customary "wavefunction" used in quantum mechanics.

In the base of complex plane waves $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, $\psi(\mathbf{r}, t)$ admits four representations related through standard Fourier transforms (FT) and their inverses:

$$\begin{array}{ccc}
\psi(\mathbf{r}, t) & \longleftrightarrow & \Psi(\mathbf{r}, \omega) \\
\updownarrow & & \updownarrow \\
\tilde{\psi}(\mathbf{k}, t) & \longleftrightarrow & \tilde{\Psi}(\mathbf{k}, \omega).
\end{array} \tag{3}$$

We will denote its ω -Fourier (time) transform by an uppercase symbol: $\Psi(\mathbf{r}, \omega)$; its k -Fourier (space) transform by a tilde: $\tilde{\psi}(\mathbf{k}, t)$, and its (\mathbf{k}, ω) -Fourier (space-time) transform by $\tilde{\Psi}(\mathbf{k}, \omega)$. We have

$$\tilde{\Psi}(\mathbf{k}, \omega) = \int \int \int \int_{-\infty}^{\infty} \psi(\mathbf{r}, t) e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} d^3r dt. \tag{4}$$

We will make use throughout of some basic properties of the FT such as

$$\begin{array}{ccc}
\partial \psi(\mathbf{r}, t) / \partial t & \longleftrightarrow & -i\omega \Psi(\mathbf{r}, \omega) \\
\nabla \psi(\mathbf{r}, t) & \longleftrightarrow & i\mathbf{k} \tilde{\psi}(\mathbf{k}, t).
\end{array} \tag{5}$$

In general, if we consider $\psi(\mathbf{r}, t)$ as an arbitrary time function, its *time analytical signal* or *time complex pre-envelope*, denoted by $\psi_+(\mathbf{r}, t)$, is defined as the function whose ω -FT is as follows:

$$\Psi_+(\mathbf{r}, \omega) = \begin{cases} 2\Psi(\mathbf{r}, \omega), & \omega > 0 \\ \Psi(\mathbf{r}, 0), & \omega = 0 \\ 0, & \omega < 0. \end{cases} \tag{6}$$

So, we consistently write

$$\psi_+(\mathbf{r}, t) \longleftrightarrow \Psi_+(\mathbf{r}, \omega). \tag{7}$$

Thus, $\psi_+(\mathbf{r}, t)$ essentially contains only the positive part of the ω -spectrum of $\psi(\mathbf{r}, t)$. As it is well known, $\psi_+(\mathbf{r}, t) = \psi(\mathbf{r}, t) + i\check{\psi}(\mathbf{r}, t)$, where $\check{\psi}(\mathbf{r}, t)$ is the Hilbert transform of $\psi(\mathbf{r}, t)$. [The simplest example is given by the complex exponential: $\exp(i\omega t) = \cos(\omega t) + i \sin(\omega t)$.]

Let us note that if $\psi(\mathbf{r}, t)$ is real, $\check{\psi}(\mathbf{r}, t)$ is real too, so we can write

$$\psi(\mathbf{r}, t) = \text{Re}[\psi_+(\mathbf{r}, t)] \equiv \text{Re}[\widehat{\psi}(\mathbf{r}, t) \exp(-i\omega_c t)]. \tag{8}$$

In analytical signal theory, the function $\widehat{\psi}(\mathbf{r}, t)$ is the so-called *time complex envelope*. The angular frequency ω_c is in principle an arbitrary parameter but, in optics or signal theory in general, the decomposition in (8) usually turns out to be useful when $\psi(\mathbf{r}, t)$ is a bandpass signal and ω_c is precisely chosen to be its carrier frequency. The physical meaning of ω_c in our present case will be explained in Section 3.

For the sake of notational simplicity and clarity of the exposition, we will assume hereafter one-dimensional wave propagation in space (say in the x direction), the generalization to three dimensions being obvious and omitted.

Now a separation between positive and negative spectral components can also be done in the spatial domain. Rather than $+$ and $-$, we will use the notation $>$ and $<$ for the k -spectrum. For instance, the “spatial analytical signal” is the function containing only the spatial frequencies $k > 0$. We denote it by $\tilde{\psi}_>$, so that

$$\tilde{\psi}_>(k, t) = \begin{cases} 2\tilde{\psi}(k, t), & k > 0 \\ \tilde{\psi}(0, t), & k = 0 \\ 0, & k < 0, \end{cases} \tag{9}$$

and we can write

$$\psi_{>}(x, t) \longleftrightarrow \tilde{\psi}_{>}(k, t). \quad (10)$$

If $\psi(x, t)$ is real, we also obtain, analogous to the first equality of Eq. (8),

$$\psi(x, t) = \text{Re}[\psi_{>}(x, t)]. \quad (11)$$

In quantum electrodynamics, the k -spectrum decomposition is frequently used —generally as a discrete sum of traveling or stationary modes— to quantify the electromagnetic field in the Heisenberg picture (see for example [3], [4]); this has nothing to do with its purpose in the present work.

To keep the space-time analogy complete, we might carry out the spatial Hilbert carrier and complex envelope decomposition analogous to (8). However, as we will see later on, these concepts are unnecessary in the k domain, so we do not need to elaborate on them.

Finally, we can write

$$\tilde{\Psi}_{>+}(k, \omega) = \begin{cases} 2\tilde{\Psi}(k, \omega), & k > 0, \omega > 0 \\ \tilde{\Psi}(0, 0), & k = 0, \omega = 0 \\ 0, & k < 0, \omega < 0, \end{cases} \quad (12)$$

so that $\psi_{>+}(x, t) \longleftrightarrow \tilde{\Psi}_{>+}(k, \omega)$, where the two arrows denote double Fourier transformation (space and time).

3 Schrödinger equation for $\widehat{\psi}_{>}$

The space-time planes of the aforementioned waves move with a constant phase velocity, v_p , determined by $v_p = \omega/k$. Thus,

$$v_p = \frac{\omega}{k} = \frac{E}{p}, \quad (13)$$

where use has been made of the quantum relationships (1) and (2) in the second equality. Hence,

$$k^2 - \omega^2/v_p^2 = 0. \quad (14)$$

This identity remains valid if multiplied for any well-behaved (complex, in general) function of k and ω . We multiply Eq. (14) precisely by the function $\tilde{\Psi}_{>+}(k, \omega)$, for reasons that will become clear later. We thus obtain the equation $(k^2 - \omega^2/v_p^2)\tilde{\Psi}_{>+}(k, \omega) = 0$. Taking its inverse (k, ω) -FT, a generic homogeneous wave equation is obtained for the doubly-analytical function $\psi_{>+}(x, t)$:

$$\frac{\partial^2 \psi_{>+}(x, t)}{\partial x^2} - \frac{1}{v_p^2} \frac{\partial \psi_{>+}(x, t)}{\partial t^2} = 0. \quad (15)$$

Indeed, a necessary (but not sufficient) condition for any such equation to meet relativistic covariance (with scalar coefficients) is to be second order both in space and time.

We must now cast Eq. (15) into its specific form. Recall that, for a free particle of rest mass (or simply “mass” [5]) m moving at a speed v , the relativistic energy and the spatial part of the four-momentum, as measured in an inertial frame, are given by

$$E = mc^2(1 - v^2/c^2)^{-1/2}; \quad p = mv(1 - v^2/c^2)^{-1/2} = c^{-1}(E^2 - m^2c^4)^{1/2}. \quad (16)$$

Using Eqs. (16) and (1), the equation $k^2 = \omega^2/c^2 - m^2c^2/\hbar^2$ follows from Eq. (14). Again, multiplying this equality through by $\tilde{\Psi}_{>+}(k, \omega)$ and taking its inverse (k, ω) -FT, we obtain

$$\frac{\partial^2 \psi_{>+}(x, t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi_{>+}(x, t)}{\partial t^2} = \frac{m^2 c^2}{\hbar^2} \psi_{>+}(x, t), \quad (17)$$

or, using Einstein’s sum convention and the relativistic notation with $ct \equiv x^0$ and $\mu \equiv mc/\hbar$,

$$(\partial^\nu \partial_\nu - \mu^2) \psi_{>+}(x^\nu) = 0. \quad (18)$$

Expression (17) or (18) is just the Klein-Gordon (KG) equation. As it is well known, the KG equation originates from the efforts to fit the theory of Quantum Mechanics in the relativistic formalism. While we have obviously used the latter, note that we have derived Eq. (17) *without* resorting to the usual quantum-mechanical prescriptions:

$$E \rightarrow i\hbar \partial/\partial t, \quad p \rightarrow -i\hbar \partial/\partial x. \quad (19)$$

It must be remarked that Eqs. (1) and (2), which have also been used, are simply “compatible” with the postulates (19) —but certainly weaker.

We now arrive at the key step. We write $\psi_{>+}(x, t)$ as a product of a “slow” time complex envelope $\hat{\psi}_{>}(x, t)$ and an time exponential oscillatory factor, as in Eq. (8). In doing so, we chose the frequency of the latter to be precisely $\omega_c = E_c/\hbar = mc^2/\hbar$, *i.e.* we associate the Hilbert carrier to the rest energy of the particle. We shall call ω_c the *latent pulsation* of $\psi_{>+}(x, t)$.

We have

$$\psi_{>+}(x, t) = \hat{\psi}_{>}(x, t) e^{-i\omega_c t}. \quad (20)$$

It is worth recalling that, as early as in 1925, de Broglie paid attention to the specific frequency $\nu_0 = m_e c^2/h$, with m_e the electron mass, for which he coined the name “proper frequency of the electron” [6]. However, not any kind of relation seems to have ever been envisioned between such “proper frequency” and the carrier frequency of an analytical signal. Operationally, the decomposition (20) is now introduced in the literature as an ansatz that yields the Schrödinger equation as the nonrelativistic limit of the KG equation (see for example [7]). In our approach, we will find that ω_c has a relevant meaning when seen as the “carrier” frequency of an analytical signal.

Replacing Eq. (20) in Eq. (17), Schrödinger’s equation is obtained for $\hat{\psi}_{>}(x, t)$:

$$\frac{\hbar^2}{2m} \frac{\partial^2 \hat{\psi}_{>}(x, t)}{\partial x^2} = -i\hbar \frac{\partial \hat{\psi}_{>}(x, t)}{\partial t}, \quad (21)$$

where a term with $\partial^2 \hat{\psi}_{>}(x, t)/\partial t^2$ has been neglected in favor of the first derivative, according to the paraxial approximation [7]. It is the wavefunction $\psi_{>+}(x, t)$, which

incorporates the rest energy, that obeys a covariant equation determined by the relativity principle, while the slow complex envelope $\widehat{\psi}_{>}(x, t)$, which has a displaced ω -spectrum, does not; Schrödinger's wavefunction $\widehat{\psi}_{>}(x, t)$ is the “slow” time complex envelope of an analytical signal $\psi_{>+}(x, t)$ with a ω -spectrum located around a very high frequency ω_c , the latent pulsation of the particle.

For a particle in a potential $V(x)$, one considers the space discretized in a set of subspaces x_i and width Δx , wherein the potential takes on constant values $V = V_i$, and infers that in the limit the equation valid for all space encompasses all the local equations and preserves the continuity of $\psi_{>+}(x, t)$ and its derivative. [This kind of *ad hoc* argument is anything but appealing theoretically, but in essence it is not so different from the approaches used in the original derivations (see for example [8]).] Namely, Eqs. (16) are now modified to the form

$$E = mc^2(1 - v^2/c^2)^{-1/2} + V; \quad p = mv(1 - v^2/c^2)^{-1/2} = c^{-1}[(E - V)^2 - m^2c^4]^{1/2}. \quad (22)$$

Proceeding as previously, there follows the equality

$$k^2 = \frac{\omega^2}{c^2} - \frac{2V}{\hbar c^2}\omega + \frac{V^2}{\hbar^2 c^2} - \frac{m^2 c^2}{\hbar^2}. \quad (23)$$

One next multiplies Eq. (23) by $\tilde{\Psi}_{>+}(k, \omega)$ and takes the inverse (k, ω) -FT, which is still trivial so long as V is not position-dependent, as assumed. This yields

$$\frac{\partial^2 \psi_{>+}(x, t)}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2 \psi_{>+}(x, t)}{\partial t^2} - i \frac{2V(x)}{\hbar c^2} \frac{\partial \psi_{>+}(x, t)}{\partial t} = \left(\frac{m^2 c^2}{\hbar^2} - \frac{V^2(x)}{\hbar^2 c^2} \right) \psi_{>+}(x, t), \quad (24)$$

which generalizes Eq. (17). Eq. (24) is the *time-dependent KG equation with a potential* V (see [9] and references therein). Of course, as announced, V had to be generalized *ad hoc* in Eq. (24) to be position-dependent.

Proceeding analogously to the free particle case, we substitute Eq. (20) in Eq. (24) and obtain, when¹ $V/(mc^2) \ll 1$, the Schrödinger equation *for the time complex envelope* $\widehat{\psi}_{>}(x, t)$:

$$\frac{\hbar^2}{2m} \frac{\partial^2 \widehat{\psi}_{>}(x, t)}{\partial x^2} + V(x) \widehat{\psi}_{>}(x, t) = -i\hbar \frac{\partial \widehat{\psi}_{>}(x, t)}{\partial t}. \quad (27)$$

¹Alternatively, Eq. (27) can be derived starting from an approximation of Eq. (23) in the first place. In the non-relativistic limit $v \ll c$, Eq. (23) can be put, after some manipulations, in the form

$$k^2 \simeq \frac{2m}{\hbar} \omega - \left(\frac{2mV}{\hbar^2} - \frac{2m^2 c^2}{\hbar^2} \right). \quad (25)$$

Multiplying Eq. (25) by $\tilde{\Psi}_{>+}(k, \omega)$ and taking its inverse (k, ω) -FT (and setting $V = V(x)$ at the end), a wave equation is obtained for $\psi_{>+}(x, t)$:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi_{>+}(x, t)}{\partial x^2} - i\hbar \frac{\partial \psi_{>+}(x, t)}{\partial t} + (V + mc^2) \psi_{>+}(x, t) = 0. \quad (26)$$

Eq. (26), which already lacks the term with the second time derivative, is an approximate version of Eq. (24) because it has been derived from the approximate equation (25). Using Eq. (20) in Eq. (26), Schrödinger's equation (27) follows exactly.

4 Average energy

We are now going to introduce the extremely important concept of *average temporal frequency* of a *real* wave $\psi(x, t)$. We must first notice that, $\psi(x, t)$ being real, $|\Psi(x, \omega)| = |\Psi(x, -\omega)|$, so it follows that $\int_{-\infty}^{\infty} \omega |\Psi(x, \omega)|^2 d\omega$ is identically zero. It is then obvious that, if we are to define an average frequency for any real function, it only makes sense to do it over the *positive* frequency range. As mentioned before, $\psi_{>+}(x, t)$ is, save for a factor of 2, a signal containing only the positive ω -frequencies of $\psi(x, t)$. So, we take

$$\langle \omega \rangle \equiv \frac{\int_L dx \int_{-\infty}^{\infty} \omega |\Psi_{>+}(x, \omega)|^2 d\omega}{\int_L dx \int_{-\infty}^{\infty} |\Psi_{>+}(x, \omega)|^2 d\omega} = N_E \int_{-\infty}^{\infty} \omega |\Psi_{>+}(x, \omega)|^2 d\omega, \quad (28)$$

where L is the normalization length (a volume, in three dimensions) and we have called $N_E \equiv [\int_L dx \int_{-\infty}^{\infty} |\Psi_{>+}(x, \omega)|^2 d\omega]^{-1} \int_L dx (\cdot)$ for the sake of notational brevity. Expression (28) would also apply with $\Psi_+(x, \omega)$ in place of $\Psi_{>+}(x, \omega)$, but we need to consider the analytical spatial signal as well, as will be seen in Section 6.

We next use Parseval's theorem [10], which states that, for two arbitrary complex functions $f_1(t)$ and $f_2(t)$,

$$\int_{-\infty}^{\infty} f_1(t) f_2^*(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(\omega) F_2^*(\omega) d\omega. \quad (29)$$

Taking $F_1(\omega) = \omega \Psi_{>+}(x, \omega)$ and $F_2(\omega) = \Psi_{>+}(x, \omega)$ in (28), and recalling that $f_1(t) = \text{FT}^{-1}\{\omega \Psi_{>+}(x, \omega)\} = i\partial\psi_{>+}(x, t)/\partial t$, we obtain

$$\langle \omega \rangle = 2\pi N_E \int_{-\infty}^{\infty} \psi_{>+}^*(x, t) i \frac{\partial \psi_{>+}(x, t)}{\partial t} dt. \quad (30)$$

Let us write Eq. (30) in terms of the temporal complex envelope $\widehat{\psi}_{>}(x, t)$, assuming a particle of rest energy E_c and latent pulsation $\omega_c = E_c/\hbar$. Replacing Eq. (20) in Eq. (30), the following relationship is obtained:

$$\langle \omega \rangle = \frac{\int_L dx \int_{-\infty}^{\infty} \widehat{\psi}_{>}^*(x, t) i \frac{\partial \widehat{\psi}_{>}(x, t)}{\partial t} dt}{\int_L dx \int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dt} + \omega_c \equiv \langle \widehat{\omega} \rangle + \omega_c. \quad (31)$$

In Eq. (31) the relationship $\int_{-\infty}^{\infty} |\Psi_{>+}(x, \omega)|^2 d\omega = 2\pi \int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dt$ has been used, which can be derived from Eq. (29).

Thus, the average frequency of $\psi_{>+}(x, t)$ is revealed to be the sum of ω_c , the latent pulsation of the rest particle, plus the *average frequency* of the spectrum of its “baseband-lying” complex envelope, denoted $\widehat{\omega}$. Multiplying Eq. (31) by \hbar , we get an expression for $\langle E \rangle$, the *total average energy*, as the sum of the rest energy of the particle and the average energy of the complex envelope:

$$\langle E \rangle = \langle \widehat{E} \rangle + E_c, \quad (32)$$

with

$$\langle \widehat{E} \rangle \equiv \hbar \langle \widehat{\omega} \rangle = \frac{\int_L dx \int_{-\infty}^{\infty} \widehat{\psi}_{>}^*(x, t) i \hbar \frac{\partial \widehat{\psi}_{>}(x, t)}{\partial t} dt}{\int_L dx \int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dt}. \quad (33)$$

As we saw in Section 3, it is the wavefunction $\widehat{\psi}_{>}(x, t)$ which appears in the Schrödinger equation. Consequently, we have been able to derive the expression (33) for the average value of the energy without using the quantum-mechanical postulates (19). In our approach, based only on the formula (1), Eq. (33) is obtained in a natural way. In fact, calling

$$\hat{E} \equiv i\hbar \frac{\partial}{\partial t}, \quad (34)$$

we can write, for the *average energy*,

$$\langle \widehat{E} \rangle = \frac{\int_L dx \int_{-\infty}^{\infty} \widehat{\psi}_{>}^*(x, t) \hat{E} \widehat{\psi}_{>}(x, t) dt}{\int_L dx \int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dt}. \quad (35)$$

The notation (34) is meant to indicate that $i\hbar \partial/\partial t$ is the “energy quantum operator,” as the result (35) strongly suggests. However, Eq. (35) is not a sufficient condition. We will return to this point in Section 7.

Note also that if the integration over all space, contained in N_E , had not been performed, $\langle E \rangle$ would have been x -dependent, which diverts from the usual consideration of the energy as a global, non-localized, characteristic of the system. Finally, it is important to realize that all the formalism developed in this section is virtually *independent of the form of the wave equation obeyed by $\widehat{\psi}_{>}$* .

5 Schrödinger’s time-independent equation

In looking for pure time harmonic solutions, one finds that the so-called Schrödinger’s time-independent equation is essentially connected to the ω -FT of Eq. (27). The latter reads, denoting $\widehat{\omega} = \widehat{E}/\hbar$ the transformed variable,

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \widehat{\Psi}_{>}(x, \widehat{\omega})}{\partial x^2} + V(x) \widehat{\Psi}_{>}(x, \widehat{\omega}) = \widehat{E} \widehat{\Psi}_{>}(x, \widehat{\omega}). \quad (36)$$

Such Fourier relation is straightforward, but is usually overlooked due to the fact that a *discrete* sum of complex time exponential corresponds to a sum of Dirac deltas in the frequency domain. Properly, one has to distinguish two cases:

(1) If $E > V(x)$, then the solutions of (36) can be simply written as

$$\widehat{\Psi}_{> \widehat{E}}(x, \widehat{\omega}) \equiv \widehat{\varphi}_{> \widehat{E}}(x), \quad (37)$$

with $\widehat{E} = \hbar \widehat{\omega}$, $\widehat{\varphi}_{> \widehat{E}}(x)$ being the complex function conveying the spatial information of the wavefunction associated to the energy \widehat{E} , which can have any value in a continuous range. With this more familiar notation, the time-independent Schrödinger equation reads

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \widehat{\varphi}_{> \widehat{E}}(x)}{\partial x^2} + V(x) \widehat{\varphi}_{> \widehat{E}}(x) = \widehat{E} \widehat{\varphi}_{> \widehat{E}}(x). \quad (38)$$

These continuous-frequency solutions are typically found in cases such as the propagation in infinite periodic media (Bloch waves), unbounded quantum-well solutions, etc.

They form a representation base for wavefunctions with a continuous energy spectrum, so that

$$\int_{-\infty}^{\infty} \widehat{\varphi}_{>\widehat{E}}^*(x) \widehat{\varphi}_{>\widehat{E}'}(x) dx = \delta(\widehat{E} - \widehat{E}'). \quad (39)$$

(2) If $E < V(x)$, one has discrete eigenfunctions of (36), that need be written as

$$\widehat{\Psi}_{>n}(x, \widehat{\omega}_n) \equiv \widehat{\varphi}_{>n}(x) \delta(\widehat{\omega} - \widehat{\omega}_n), \quad (40)$$

the corresponding eigenvalues being $\widehat{E}_n = \hbar \widehat{\omega}_n$. Then, there follows

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \widehat{\varphi}_{>n}(x)}{\partial x^2} + V(x) \widehat{\varphi}_{>n}(x) = \widehat{E} \widehat{\varphi}_{>n}(x). \quad (41)$$

It is this case that corresponds to the true *stationary* waves, made up of pairs of identical waves propagating in both directions, $>$ and $<$. In fact, since (41) also holds for $\widehat{\varphi}_{<n}(x) = \widehat{\varphi}_{>n}^*(x)$, adding both equations yields

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \widehat{\varphi}_n(x)}{\partial x^2} + V(x) \widehat{\varphi}_n(x) = \widehat{E}_n \widehat{\varphi}_n(x), \quad (42)$$

with is the true time-independent *stationary* Schrödinger's equation. Eq. (42) is the wave equation that applies to one-dimensional cases such as an infinite potential well, the bounded wavefunctions of a finite potential well, the harmonic oscillator, etc. Indeed, it is well known, and can be proved algebraically with no difficulty, that the eigenfunctions corresponding to discrete non-degenerate eigenvalues are necessarily *real-valued*, the mentioned cases being typical examples.

The corresponding Schrödinger n -eigenfunctions are

$$\widehat{\psi}_{>n}(x, t) = \widehat{\varphi}_{>n}(x) e^{-i\widehat{E}_n t/\hbar}. \quad (43)$$

Now recalling Eq. (20), the following result is obtained:

$$\psi_{n>+}(x, t) = \widehat{\varphi}_{>n}(x) e^{-i\widehat{E}_n t/\hbar} e^{-iE_c t/\hbar} \equiv \widehat{\varphi}_{>n}(x) e^{-i\widehat{E}_n t/\hbar}. \quad (44)$$

We thus find that the oscillation frequencies have the form $\omega_n = \omega_c + \widehat{\omega}_n = E_c/\hbar + \widehat{E}_n/\hbar$. This is, each ω_n contains the rest energy E_c in addition to the familiar energy eigenvalues of the Schrödinger equation, \widehat{E}_n . [This is, naturally, in agreement with the result (32).]

6 Average momentum

As in Section 4, we expect that, in view of Eq. (2), the average momentum in the quantum state $\psi(x, t)$ can be obtained from the average wavevector of the wavefunction, $\langle k \rangle$. As is the case of the temporal frequency ω , the average k of any real-valued signal is strictly zero, so it only makes sense to compute $\langle k \rangle$ with the spatial *analytical* signal. In fact, we have already adopted this choice with $\psi_{>+}(x, t)$. We can write

$$\langle k \rangle \equiv \frac{\int_{-\infty}^{\infty} k |\widetilde{\psi}_{>+}(k, t)|^2 dk}{\int_{-\infty}^{\infty} |\widetilde{\psi}_{>+}(k, t)|^2 dk} = \frac{\int_{-\infty}^{\infty} k |\widetilde{\psi}_{>}(k, t)|^2 dk}{\int_{-\infty}^{\infty} |\widetilde{\psi}_{>}(k, t)|^2 dk} \equiv N_k \int_{-\infty}^{\infty} k |\widetilde{\psi}_{>}(k, t)|^2 dk, \quad (45)$$

with $N_k \equiv [\int_{-\infty}^{\infty} |\hat{\psi}_{>}(k, t)|^2 dk]^{-1}$. In Eq. (45) we have chosen to express $\langle k \rangle$ in terms of the complex time envelope $\hat{\psi}_{>}(k, t)$, rather than the analytical time signal $\tilde{\psi}_{>+}(k, t)$, because we found in Section 3 that it is the former that appears in the standard Schrödinger equation (27). Unlike the ω -spectrum, there is no Hilbert frequency in the space domain since the momentum of the rest particle is zero. Consequently, the k spectrum of $\psi(x, t)$ is “low band” and there is no need to consider any factorization analogous to Eq. (20).

We use again Parseval’s theorem (28) with $t \rightarrow x$ and $\omega \rightarrow k$, applied to $F_1(k) = k\hat{\psi}_{>}(k, t)$ and $F_2(k) = \hat{\psi}_{>}(k, t)$. Using $f_1(x) = k\text{-FT}^{-1}\{k\hat{\psi}_{>}(k, t)\} = -i\partial\hat{\psi}_{>}(x, t)/\partial x$, we get

$$\langle k \rangle = \frac{-\int_{-\infty}^{\infty} \hat{\psi}_{>}^*(x, t) i \frac{\partial \hat{\psi}_{>}(x, t)}{\partial x} dx}{\int_{-\infty}^{\infty} |\hat{\psi}_{>}(x, t)|^2 dx}. \quad (46)$$

Multiplying Eq. (46) by \hbar , using Eq. (2), and calling

$$\hat{p} \equiv -i\hbar \frac{\partial}{\partial x}, \quad (47)$$

we can write, for the *average momentum*,

$$\langle p \rangle = \frac{\int_{-\infty}^{\infty} \hat{\psi}_{>}^*(x, t) \hat{p} \hat{\psi}_{>}(x, t) dx}{\int_{-\infty}^{\infty} |\hat{\psi}_{>}(x, t)|^2 dx}. \quad (48)$$

Eq. (48) has the same form as the standard expression for the average value of the quantum-mechanical momentum when the system is in the state $\hat{\psi}_{>}(x, t)$. Once again, this result has been derived without resorting to the postulates (19). That $-i\hbar\partial/\partial x$ is truly the “quantum momentum operator” will be seen in the next section.

7 Measurement probability

In Sections 4 and 6 we have obtained the expressions of the average energy and momentum, Eqs. (33) and (48), respectively. Although these results are promising, the much more general probability postulate remains to be justified. With this purpose, we consider the Schrödinger wavefunction of an arbitrary quantum state, expanded in the base of stationary Schrödinger n -eigenfunctions, assumed discrete, Eq. (43):

$$\hat{\psi}_{>}(x, t) = \sum_n a_n \hat{\varphi}_{>n}(x) e^{-i\hat{E}_n t/\hbar}, \quad (49)$$

Making use of the formalism developed in Section 4, the r -th moment of the complex envelope frequency is found to be

$$\langle \hat{\omega}^r \rangle = \frac{\int_L dx \int_{-\infty}^{\infty} \hat{\Psi}_{>}^*(x, \omega) \omega^r \hat{\Psi}_{>}(x, \omega) d\omega}{\int_L dx \int_{-\infty}^{\infty} |\hat{\Psi}_{>}(x, \omega)|^2 d\omega}. \quad (50)$$

Using Eq. (29), recalling the relation $\omega^r \hat{\Psi}(x, \omega) \leftrightarrow i^r \partial^r \psi(x, t)/\partial t^r$, and multiplying Eq. (50) by \hbar^r , we obtain

$$\langle \widehat{E}^r \rangle = \hbar^r \langle \widehat{\omega}^r \rangle = \frac{\int_L dx \int_{-\infty}^{\infty} \widehat{\psi}_{>}^*(x, t) (i\hbar)^r \frac{\partial^r \widehat{\psi}_{>}(x, t)}{\partial t^r} dt}{\int_L dx \int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dt}. \quad (51)$$

Using the expansion in the base (49), the orthonormality condition $\int_L \widehat{\varphi}_{>n}^*(x) \widehat{\varphi}_{>m}(x) dx = \delta_{nm}$ and the normalization $\sum_n |a_n|^2 = 1$, we obtain

$$\langle \widehat{E}^r \rangle = \hbar^r \langle \widehat{\omega}^r \rangle = \sum_n |a_n|^2 \widehat{E}_n^r. \quad (52)$$

It is obvious that, if $|a_n|^2$ were the probability of measuring the energy \widehat{E}_n , then the result (52) would immediately follow. However, it is *the sufficient condition* that we must prove; i.e., that, if the moment formula (52) holds, then $|a_n|^2$ is the *probability* of measuring the energy \widehat{E}_n (above E_c , actually) when the particle is in the state $\widehat{\psi}_{>}(x, t)$. This is indeed so and we give the proof next.

In order to prove that $|a_n|^2$ is the probability of measuring \widehat{E}_n —or, in the continuous case, that $|a(\widehat{E})|^2$ is the probability density for \widehat{E} —, we will make use of the so-called *characteristic function* of a random variable, which, for a continuous variable with probability density $P(\widehat{E})$, is defined as [11]

$$G(s) \equiv \int_{-\infty}^{\infty} P(\widehat{E}) e^{is\widehat{E}} d\widehat{E}. \quad (53)$$

Now, $\exp(is\widehat{E}) = \sum_{r=0}^{\infty} (is)^r \widehat{E}^r / r!$, which, replaced in (53) yields

$$G(s) = \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \langle \widehat{E}^r \rangle. \quad (54)$$

For a discrete probability distribution, the characteristic function is defined as $G(s) \equiv \sum_n P_n e^{isE_n}$, and the same result (54) is obtained.

In our case, we have

$$G(s) = \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \langle \widehat{E}^r \rangle = \begin{cases} \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \sum_n |a_n|^2 \widehat{E}_n^r & \text{(discrete)} \\ \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \int_{\widehat{E}} |a(\widehat{E})|^2 \widehat{E}^r d\widehat{E} & \text{(continuous).} \end{cases} \quad (55)$$

For a continuous distribution, the inverse transform of (53) then reads

$$\begin{aligned} P(\widehat{E}) &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} G(s) e^{-is\widehat{E}} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{(is)^r}{r!} \left[\int_{\widehat{E}} |a(\widehat{E})|^2 \widehat{E}^r d\widehat{E} \right] e^{-is\widehat{E}} ds \\ &= \frac{1}{2\pi} \int_{\widehat{E}'} d\widehat{E}' |a(\widehat{E}')|^2 \int_{-\infty}^{\infty} e^{-is\widehat{E}} \underbrace{\sum_{r=0}^{\infty} \frac{1}{r!} \widehat{E}'^r (is)^r}_{\exp(is\widehat{E}')} ds \\ &= \frac{1}{2\pi} \int_{\widehat{E}'} d\widehat{E}' |a(\widehat{E}')|^2 2\pi \delta(\widehat{E}' - \widehat{E}) = |a(\widehat{E})|^2, \quad \text{Q.E.D.} \end{aligned} \quad (56)$$

The discrete distribution can be treated in the same framework by putting $|a(\widehat{E})|^2 = \sum_n |a_n|^2 \delta(\widehat{E} - \widehat{E}_n)$. The corresponding result is obtained straightforwardly.

It is interesting to note that, in the derivation of expression (52), the quotient $\int_{-\infty}^{\infty} dt / \int_{-\infty}^{\infty} dt$ arises, leading to a mathematical difficulty. However, since the result (52) is known to be *correct*, it appears we should decide that $\int_{-\infty}^{\infty} dt / \int_{-\infty}^{\infty} dt = 1$. This problem indeed resembles that of the normalization of the wavefunction of a free particle in an infinite volume. In the present case, we may argue that the integrals can be thought of as to be extended over the interval $[-T, T]$, with T *very large* (but not infinite) as $\widehat{\psi}_{>}(x, t)$ is 0 for all practical purposes at remote times.

Again, we have derived this quantum-mechanical postulate by using only Eq. (2). It is also clear that $i\hbar\partial/\partial t$ should be considered as the energy operator, as we conjectured in Section 4. Naturally, the quantum-mechanical formula for the average energy is the particular case of Eq. (52) with $r = 1$.

It is trivial to carry out a similar derivation for a continuous energy spectrum,

$$\widehat{\psi}_{>}(x, t) = \int_{\widehat{E}} a(\widehat{E}) \widehat{\varphi}_{>\widehat{E}}(x) e^{-i\widehat{E}t/\hbar} d\widehat{E}, \quad (57)$$

with $\int_L \widehat{\varphi}_{>\widehat{E}}^*(x) \widehat{\varphi}_{>\widehat{E}'}(x) dx = \delta(\widehat{E} - \widehat{E}')$.

The derivation for the momentum follows the same guidelines. Let $p_j = \hbar k_j$ be a set of momentum eigenvalues. We write

$$\widehat{\psi}_{>}(x, t) = \sum_j b_j(t) e^{ik_j x}. \quad (58)$$

Generalizing Eq. (46), we have

$$\begin{aligned} \langle p^r \rangle &= \hbar^r \langle k^r \rangle = \hbar^r \frac{\int_{-\infty}^{\infty} k^r |\widehat{\psi}_{>}(k, t)|^2 dk}{\int_{-\infty}^{\infty} |\widehat{\psi}_{>}(k, t)|^2 dk} = \frac{\int_{-\infty}^{\infty} \widehat{\psi}_{>}^*(x, t) (-i\hbar)^r \frac{\partial^r \widehat{\psi}_{>}(x, t)}{\partial x^r} dx}{\int_{-\infty}^{\infty} |\widehat{\psi}_{>}(x, t)|^2 dx} \\ &= \sum_j |b_j(t)|^2 (\hbar k_j)^r, \end{aligned} \quad (59)$$

so $|b_j(t)|^2$ is the probability of measuring the momentum $p_j = \hbar k_j$ and $i\hbar\partial/\partial x$ is found to be the momentum operator, as presumed.

As a final point, we see that the observation made at the end of Section 4 is confirmed: The formalism that allows to derive the form of the quantum operators and the probability formulas for the momentum and energy, relies purely on the Fourier theory and statistics. Although the specific “eigenstates of the Schrödinger equation,” for example, have been brought up in the discussion, a quick review of the procedure reveals that the derivation is not really subordinated, at a deep level, to the *specific form* of the wave equation.

8 Concluding remarks

As it has been seen in the preceding sections, the standard symbol “ $\psi(x, t)$ ” for the quantum wavefunction appears very scarcely in this article. On the contrary, the unfamiliar and cumbersome notations $\widehat{\psi}_{>}(x, t)$ and $\psi_{>+}(x, t)$ have been used abundantly.

Indeed, this deliberate typographical waste has been the price to pay to keep clear at all times what the so-called Schrödinger's wavefunction is, avoiding any confusion with other related but different functions.

The intriguing resemblance of the analytical signal theory with the formalism of Schrödinger's wavefunction was the clue that motivated this work. The form of Schrödinger's complex wavefunction, $\widehat{\psi}_{>}(x, t)$ in our notation, suggested that it might in fact be the temporal analytical part of some real-valued space-time function containing a rapid oscillation. We have denoted $\psi_{>+}(x, t)$ its corresponding analytical signal, whose Hilbert carrier is the ultrafast oscillation corresponding to the rest energy of the particle.

As far as Quantum mechanics is concerned, we have started from the two very basic relationships (1) and (2) that associate a particle to a wave. Other than that, we have only applied wave theory in strict terms, without resorting to any other quantum postulate.

Note that nothing indicated *a priori* that these functions should be spatially analytical as well, since the equations would equally apply to the corresponding functions without the $>$ subscript. However, in order to prove the momentum postulate (Section 6), we have anticipated from the very start that the complex wavefunctions need also be analytical in the spatial spectrum.

Using wave mechanics exclusively, we have been able to give a plausible explanation for the fundamental expressions of the energy and momentum quantum operators, Eq. (19) (only for the one-particle case). To accomplish this, we have first dealt with the computation of the average values, and then derived the general postulate of the measurement probability for the energy and the momentum. This simultaneously yielded the form of the operators. However, of course, no light can be shed on the part of the postulate concerning *the collapse of the wavefunction*.

In this work, no attempt has certainly been made to deal, for example, with multiparticle systems, continuous systems (field quantization) or Hamiltonian-Lagrangian approaches. We have focused on a simple quantum system, and a collection of surprising results have been obtained by simply looking at the complex wavefunction from a fresh perspective. Some non-relativistic applications of the formalism and further development of the theory, including the phase problem, will be the object of future work.

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